

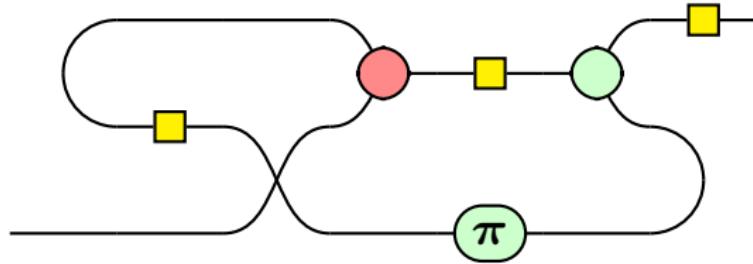
Scalable Graphical Quantum Reasoning

Titouan Carette, joint work with Dominic Horsman and Simon Perdrix.

LORIA, équipe MOCQUA, Nancy

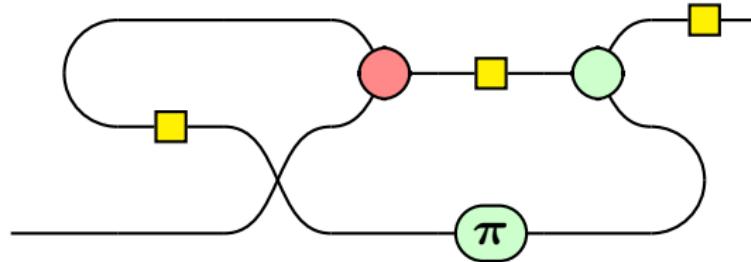
June 7, 2019

ZX-calculus I: The syntax



ZX-diagrams are lax quantum circuits:

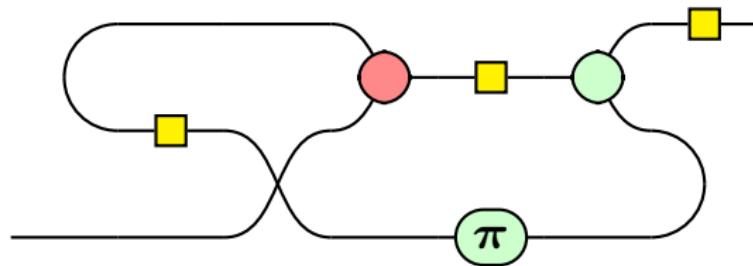
ZX-calculus I: The syntax



ZX-diagrams are lax quantum circuits:

- ⊖ Arbitrary complex matrices instead of unitaries.

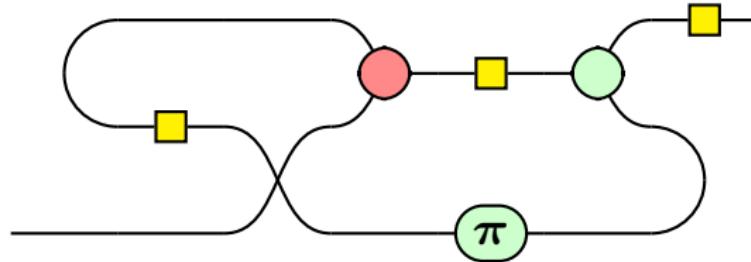
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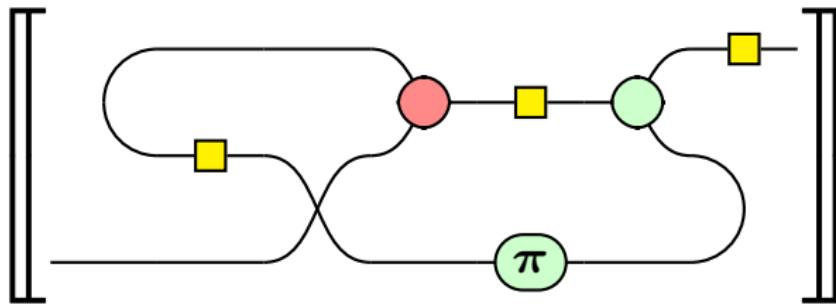
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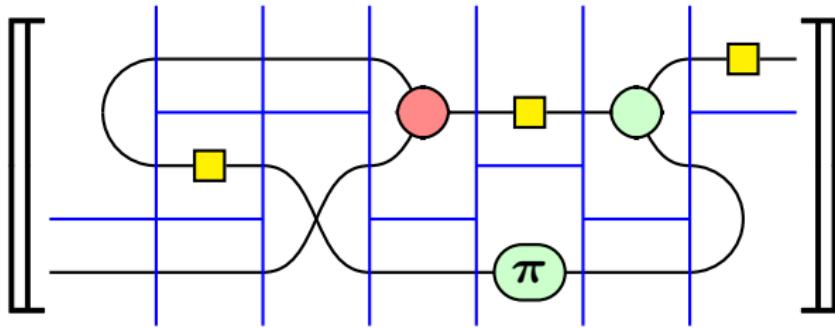
ZX-diagrams are lax quantum circuits:

- ⊖ Arbitrary complex matrices instead of unitaries.
- ⊖ Post-selection.
- ⊖ The wires are flexible.

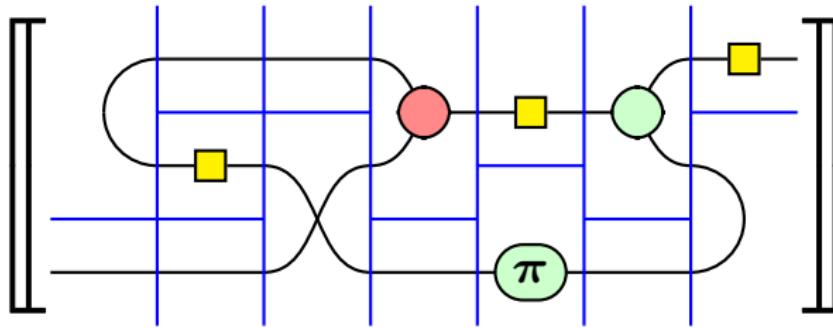
ZX-calculus II: The semantic



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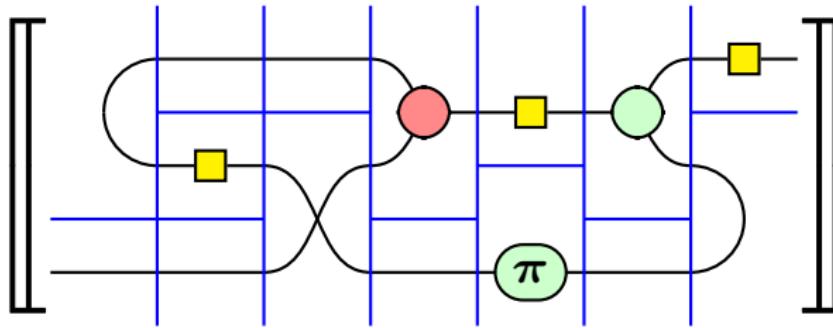


ZX-calculus II: The semantic



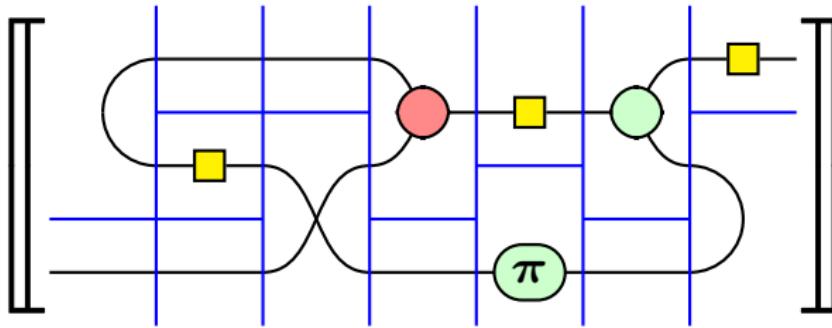
$$[H \otimes (1 \ 0 \ 0 \ 1)]$$

ZX-calculus II: The semantic



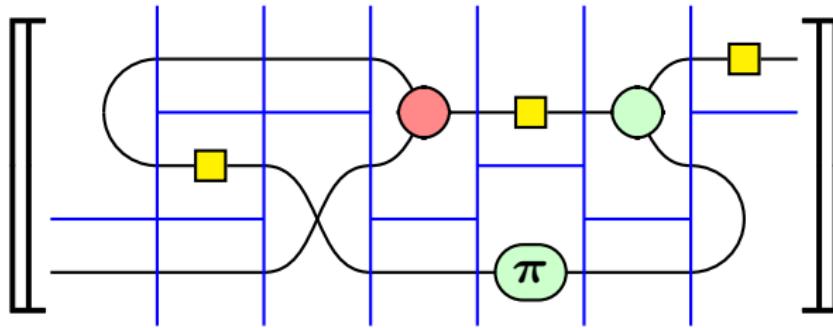
$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right]$$

ZX-calculus II: The semantic



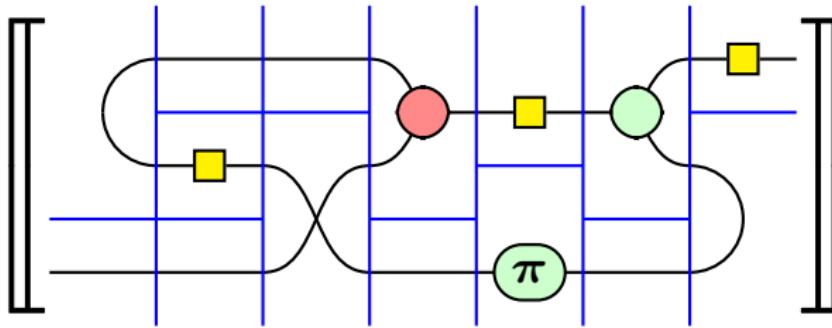
$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right] \circ [H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}]$$

ZX-calculus II: The semantic



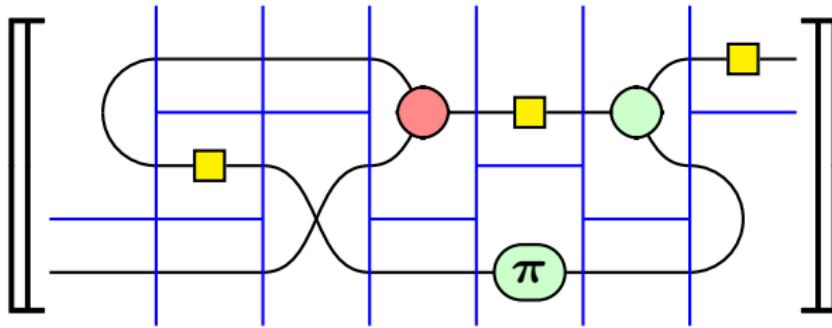
$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right] \circ [H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}] \circ \left[\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes I \right]$$

ZX-calculus II: The semantic



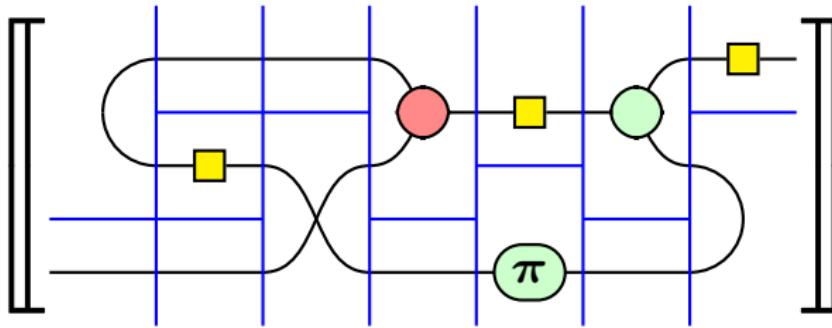
$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right] \circ \left[H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} \right] \circ \left[\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes I \right] \circ \left[I \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right]$$

ZX-calculus II: The semantic



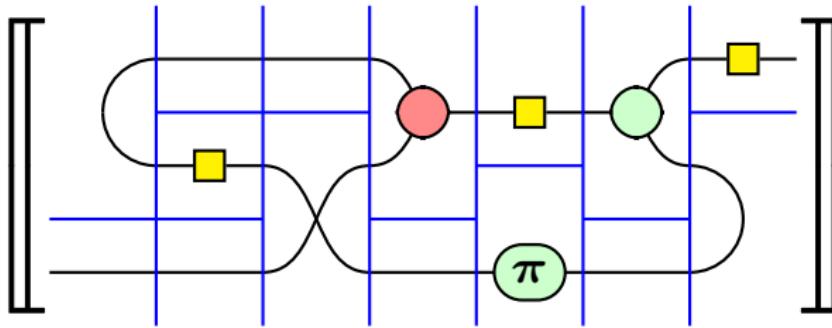
$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right] \circ [H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix}] \circ \left[\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes I \right] \circ \left[I \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \circ [I \otimes H \otimes I]$$

ZX-calculus II: The semantic



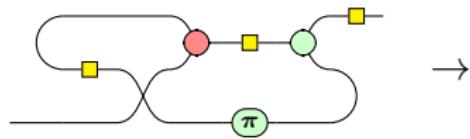
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ZX-calculus II: The semantic

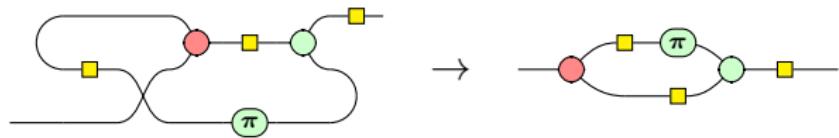


$$[H \otimes (1 \ 0 \ 0 \ 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \ 0 \ 0 \ 1) \right] \circ \left[H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} \right] \circ \left[\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes I \right] \circ \left[I \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] \circ [I \otimes H \otimes I] \circ \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \otimes I \right] = ?$$

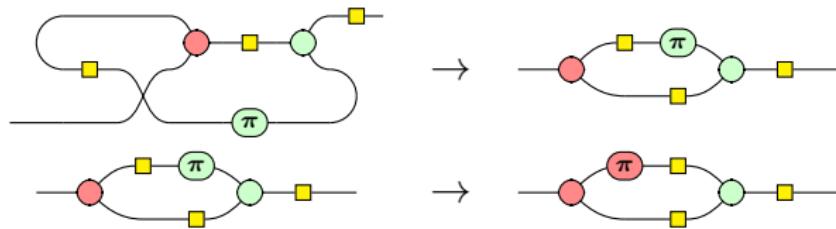
ZX-calculus III: The rules



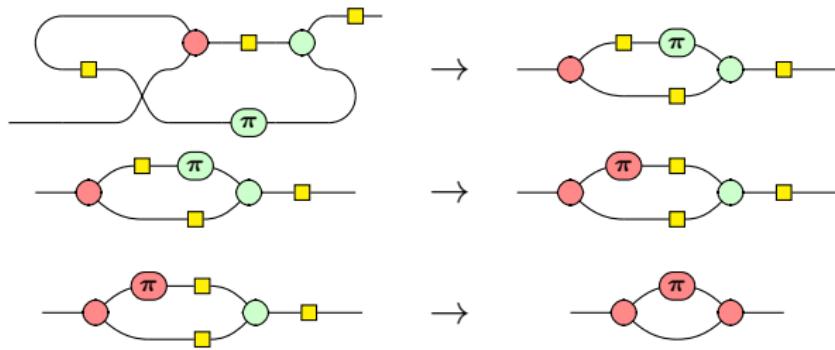
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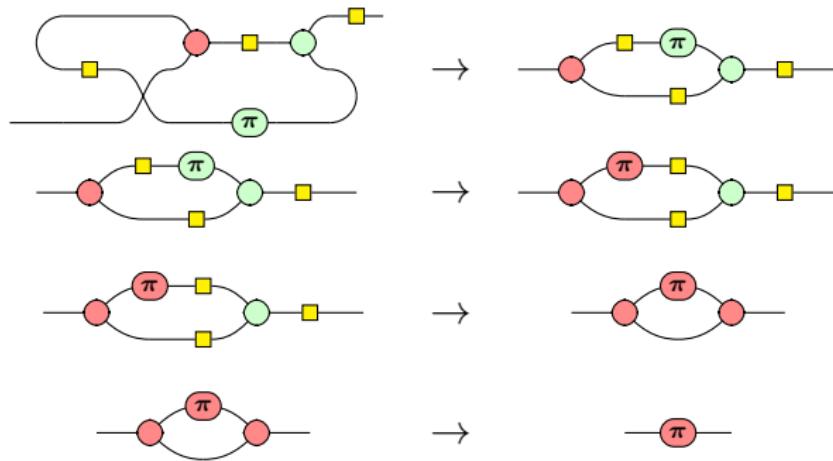
ZX-calculus III: The rules



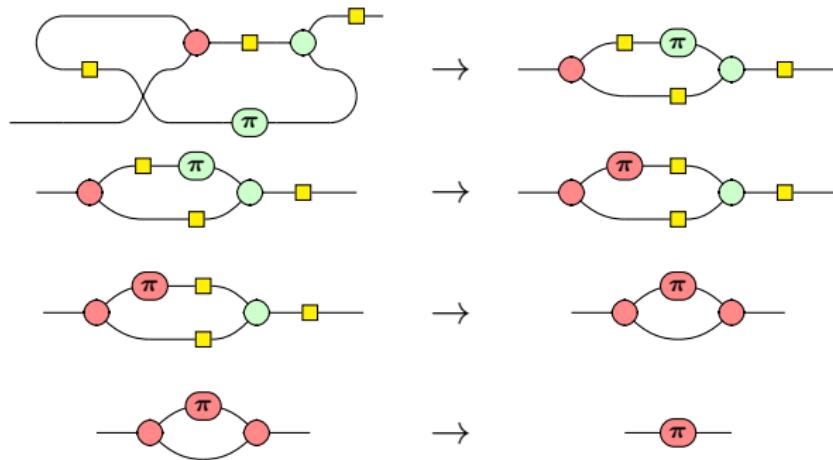
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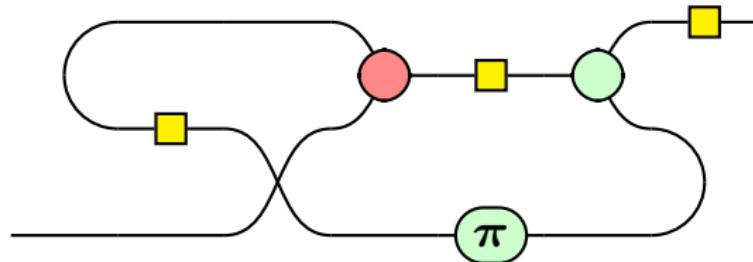


ZX-calculus III: The rules



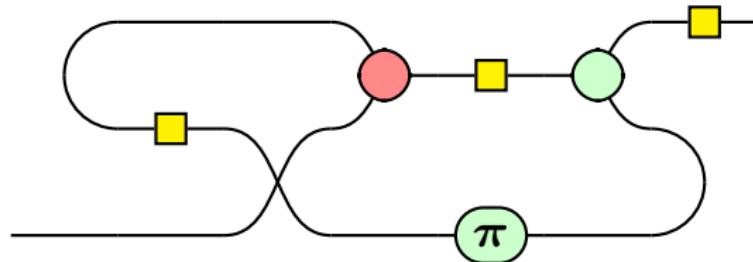
$$[\![\text{---} \circlearrowleft \text{---}]\!] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

ZX-calculus IV: Conclusion



The ZX-calculus provides:

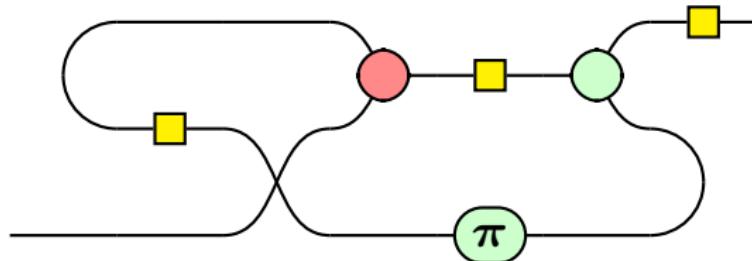
ZX-calculus IV: Conclusion



The ZX-calculus provides:

- ⊕ Intuitive graphical calculus.

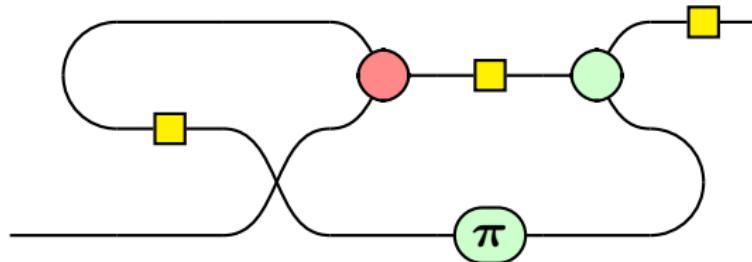
ZX-calculus IV: Conclusion



The ZX-calculus provides:

- ⊕ Intuitive graphical calculus.
- ⊕ Complete equational theory for any number of qubits.

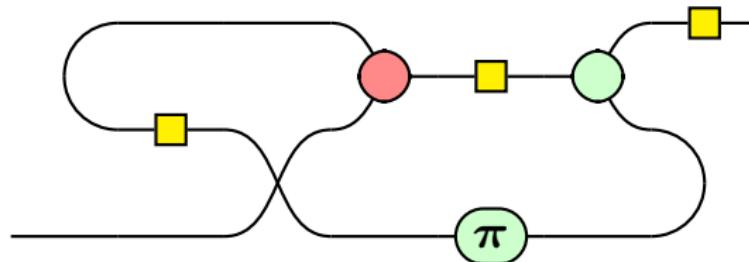
ZX-calculus IV: Conclusion



The ZX-calculus provides:

- ⊕ Intuitive graphical calculus.
- ⊕ Complete equational theory for any number of qubits.
- ⊕ Compact way to represent information.

ZX-calculus IV: Conclusion



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- ⊕ Intuitive graphical calculus.
- ⊕ Complete equational theory for any number of qubits.
- ⊕ Compact way to represent information.

We can be even more compact while scaling up the number of qubits!

$\mathcal{S}\mathbb{Z}\mathbb{X}$ -calculus I: Divide and Gather

a_b —— a_b

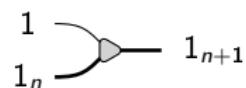
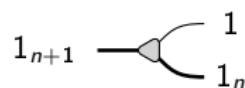
$\mathcal{S}\mathcal{Z}\mathcal{X}$ -calculus I: Divide and Gather

$$a_b \text{ --- } a_b \qquad 1_2 \text{ --- } 1_2 \neq \begin{array}{c} 1_1 \text{ --- } 1_1 \\ + \\ 1_1 \text{ --- } 1_1 \end{array}$$

\mathcal{SZX} -calculus I: Divide and Gather

$$a_b \text{ --- } a_b$$

$$1_2 \text{ --- } 1_2 \neq \begin{matrix} 1_1 & \text{---} & 1_1 \\ + & & + \\ 1_1 & \text{---} & 1_1 \end{matrix}$$



\mathcal{SZX} -calculus I: Divide and Gather

$$a_b \text{ — } a_b$$

$$1_2 \text{ — } 1_2 \neq \begin{matrix} 1_1 & \text{——} & 1_1 \\ + & & + \\ 1_1 & \text{——} & 1_1 \end{matrix}$$

$$1_{n+1} \text{ — } \begin{matrix} 1 \\ 1_n \end{matrix}$$

$$\begin{matrix} 1 \\ 1_n \end{matrix} \text{ — } 1_{n+1}$$

$$\text{—— } = \text{—— }$$

$$\text{—— } = \text{—— }$$

\mathcal{SZX} -calculus II: The rewiring theorem

Theorem:

Let $\omega \in \mathbb{W}[a, b]$ and $\omega' \in \mathbb{W}[c, d]$:

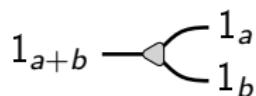
$$\mathbb{W} \vdash \omega = \omega' \Leftrightarrow a = c \text{ and } b = d$$

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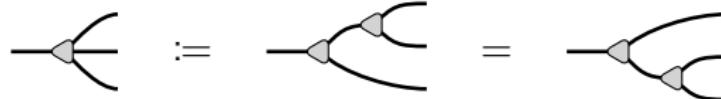
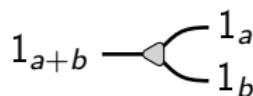


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\mathcal{SZX} -calculus III: The big generators

$$\text{Diagram showing two equivalent circuit configurations separated by an equals sign. The left configuration has a horizontal wire with a yellow square at its start, followed by a grey diamond-shaped node with two curved output lines. The right configuration shows the same components but with the yellow square moved to the end of the horizontal wire, and a small black square is placed on the lower curved output line.}$$

\mathcal{SZX} -calculus III: The big generators

$$\text{---} \square \text{---} = \text{---} \square \text{---}$$

$$\text{---} \alpha::\beta \text{---} = \text{---} \alpha \text{---} \beta \text{---}$$

$$\text{---} \alpha::\beta \text{---} = \text{---} \alpha \text{---} \beta \text{---}$$

\mathcal{SZX} -calculus IV: Completeness

$$\llbracket _ \rrbracket_s := (id_n, 1_n, 1_n)$$

SZX-calculus IV: Completeness

$$[-]_s := (id_n, 1_n, 1_n)$$

$$[\![\text{---}\blacksquare\text{---}]\!]_s := \left(\frac{1}{\sqrt{2^n}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \bullet y} |y\rangle\langle x|, 1_n, 1_n \right)$$

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$$\llbracket \text{---} \rrbracket_s := (id_n, 1_n, 1_n)$$

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$$\llbracket \text{---} \curvearrowleft \rrbracket_s := (id_{n+1}, 1_{n+1}, 1+1_n)$$

\mathcal{SZX} -calculus IV: Completeness

$$\llbracket \text{---} \rrbracket_s := (id_n, 1_n, 1_n)$$

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$$\llbracket \text{---} \text{c} \rrbracket_s := (id_{n+1}, 1_{n+1}, 1+1_n)$$

$$\llbracket \text{---} \text{b} \rrbracket_s := \left(\sum_{x \in \{0,1\}^n} e^{ix \bullet \alpha} |x^k\rangle\langle x^\ell|, k_n, \ell_n \right)$$

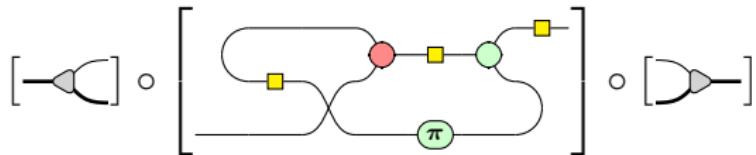
\mathcal{SZX} -calculus IV: Completeness

$$\llbracket \text{---} \rrbracket_s := (id_n, 1_n, 1_n)$$

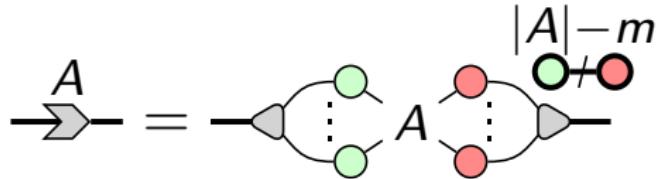
$$\llbracket \text{---} \square \rrbracket_s := (\frac{1}{\sqrt{2^n}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \bullet y} |y\rangle\langle x|, 1_n, 1_n)$$

$$\llbracket \text{---} \wedge \rrbracket_s := (id_{n+1}, 1_{n+1}, 1+1_n)$$

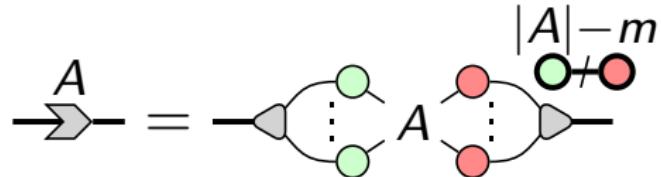
$$\llbracket \text{---} \alpha \rrbracket_s := (\sum_{x \in \{0,1\}^n} e^{ix \bullet \alpha} |x^k\rangle\langle x^\ell|, k_n, \ell_n)$$



\mathcal{SZX} -calculus V: Matrices

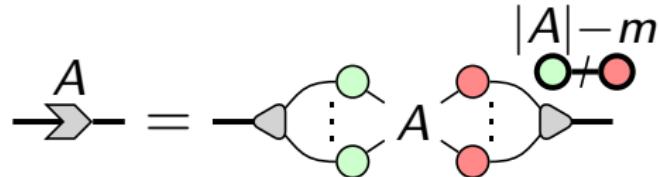


\mathcal{SZX} -calculus V: Matrices

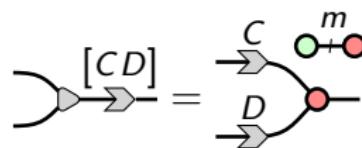
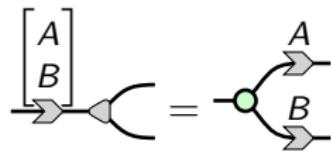
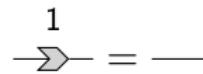
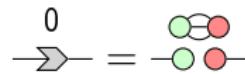


$$\forall A \in \mathbb{F}_2^{m \times n}, \left[\begin{array}{c} A \\ \rightarrow \end{array} \right]_s = (|x\rangle \mapsto |Ax\rangle, 1_n, 1_m)$$

\mathcal{SZX} -calculus V: Matrices



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SZX -calculus VI: Properties of matrices

$$\begin{array}{c} A \\ \nearrow \searrow \end{array} = \begin{array}{c} A \\ \nearrow \searrow \\ A \end{array}$$

$$\begin{array}{c} A \\ \nearrow \end{array} = \text{---}$$

$$\begin{array}{c} \pi v \\ \nearrow \end{array} A = A \begin{array}{c} \pi \\ \nearrow \end{array} v$$

$$\begin{array}{c} \eta \\ \nearrow \searrow \\ \text{---} \end{array} A = \begin{array}{c} A \\ \nearrow \searrow \\ A \end{array}$$

$$\begin{array}{c} n \\ \nearrow \searrow \\ \text{---} \end{array} A = \begin{array}{c} m \\ \nearrow \searrow \\ \text{---} \end{array}$$

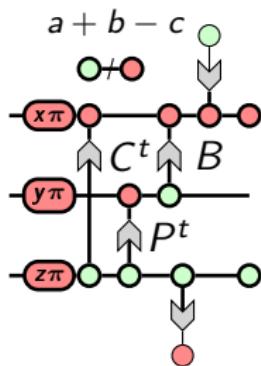
$$\begin{array}{c} A \\ \nearrow \end{array} \pi u = \pi A^t u \begin{array}{c} A \\ \nearrow \end{array}$$

$$\begin{array}{c} A \\ \nearrow \end{array} \begin{array}{c} m \\ \nearrow \searrow \\ \text{---} \end{array} = \begin{array}{c} n \\ \nearrow \searrow \\ \text{---} \end{array} A^t$$

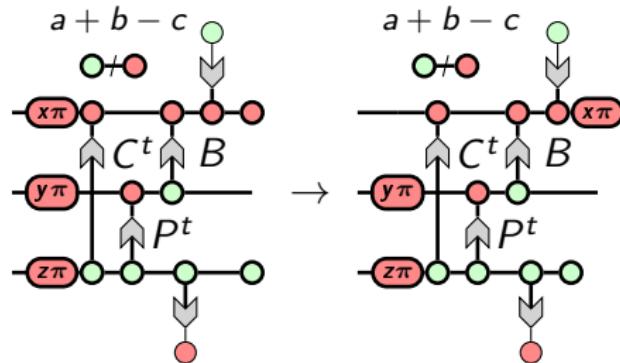
$$\begin{array}{c} A \\ \nearrow \end{array} \begin{array}{c} m \\ \nearrow \searrow \\ \text{---} \end{array} = \begin{array}{c} A+B \\ \nearrow \end{array}$$

$$\begin{array}{c} A \\ \nearrow \end{array} C = CA$$

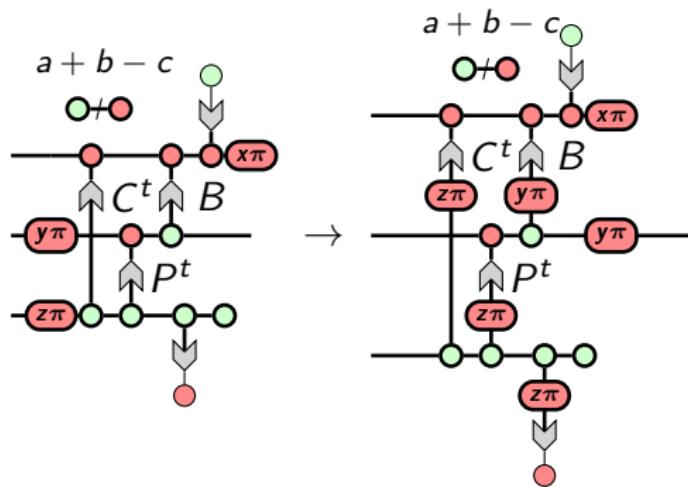
\mathcal{SZX} -calculus VII: An application



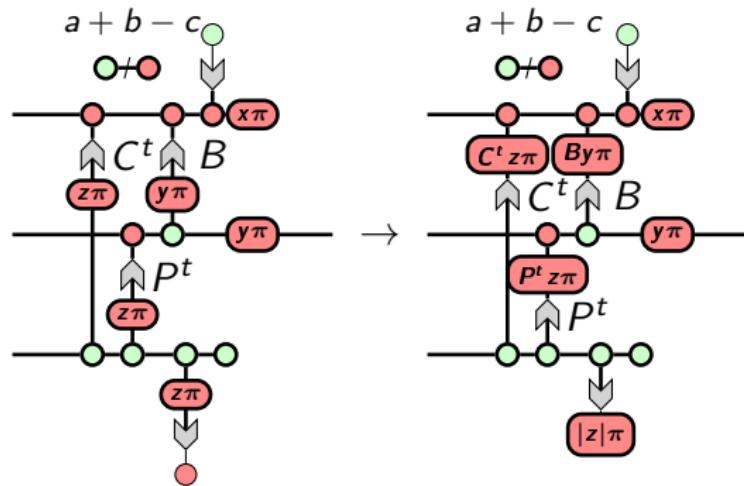
\mathcal{SZX} -calculus VII: An application



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