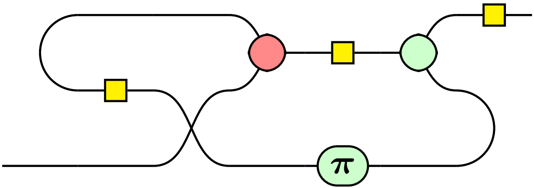
Scalable Graphical Quantum Reasoning

Titouan Carette, joint work with Dominic Horsman and Simon Perdrix.

LORIA, équipe MOCQUA, Nancy

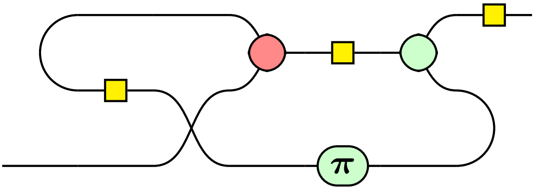
June 7, 2019

ZX-calculus I: The syntax



ZX-diagrams are lax quantum circuits:

ZX-calculus I: The syntax



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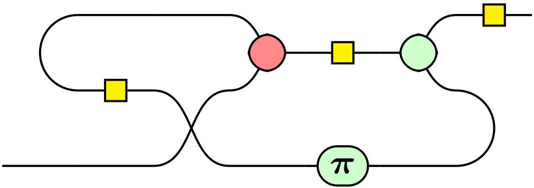

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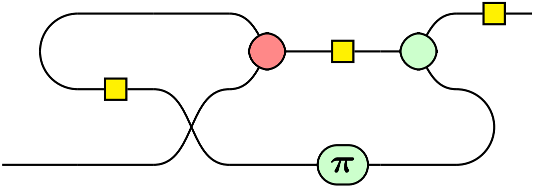
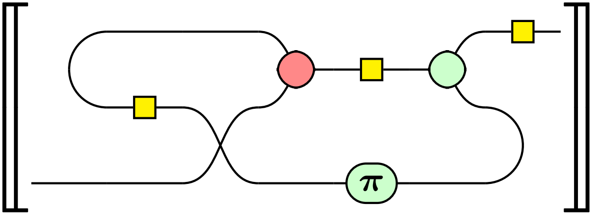


ZX-diagrams are lax quantum circuits:

- Arbitrary complex matrices instead of unitaries.
- Post-selection.
- The wires are flexible.

ZX-calculus II: The semantic



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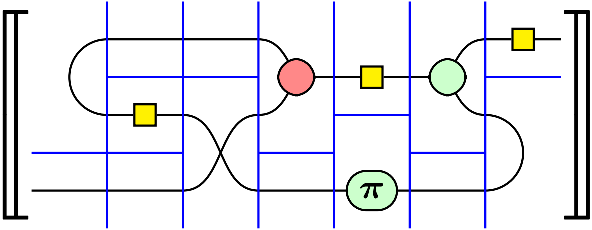

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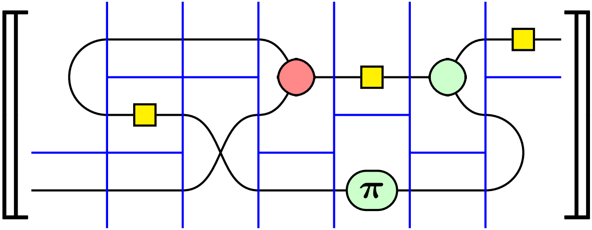

$[H \otimes (1 \quad 0 \quad 0 \quad 1)]$

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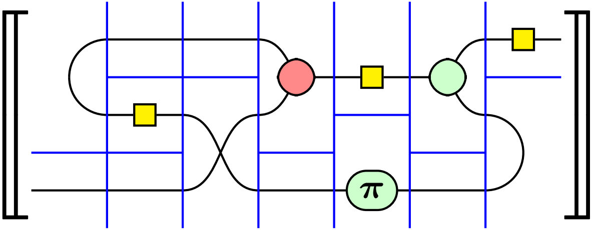

$$[H \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}] \circ \left[\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix} \right]$$

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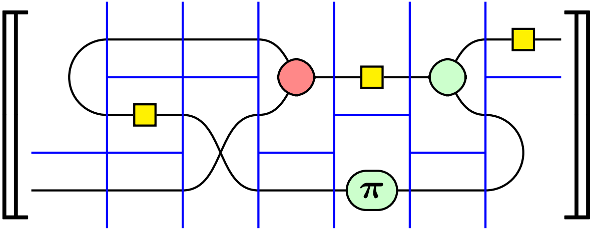

$$\left[H\otimes\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\right]\circ\left[\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 1\end{pmatrix}\otimes\begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\right]\circ\left[H\otimes\begin{pmatrix}1 & 0\\0 & e^{i\pi}\end{pmatrix}\right]$$

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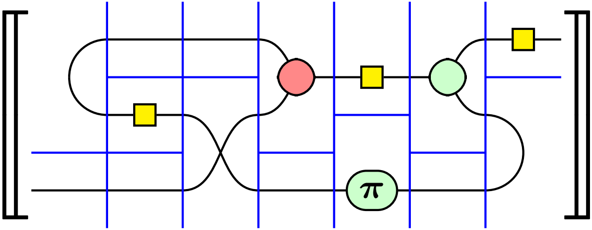

$$[H \otimes (1 \quad 0 \quad 0 \quad 1)] \circ \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1 \quad 0 \quad 0 \quad 1) \right] \circ \left[H \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} \right] \circ \left[\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \otimes I \right]$$

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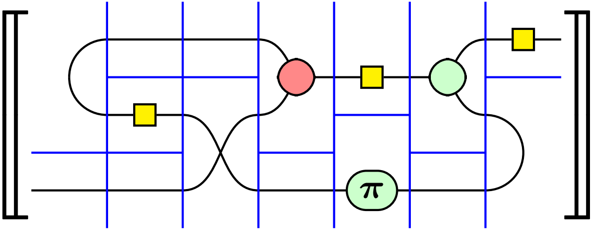

$$\left[H \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\right] \circ \left[\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\right] \circ \left[H \otimes \begin{pmatrix}1 & 0\\0 & e^{i\pi}\end{pmatrix}\right] \circ \left[\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{pmatrix} \otimes I\right] \circ \left[I \otimes \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\right]$$

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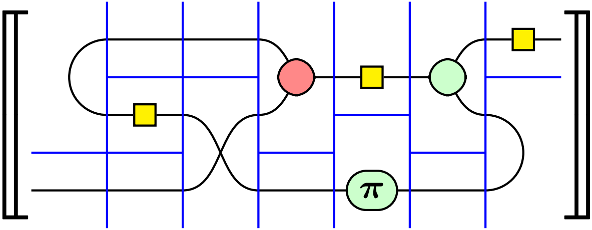

$$[H \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}] \circ \left[\begin{pmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0 & 1\end{pmatrix}\right] \circ \left[H \otimes \begin{pmatrix}1 & 0\\0 & e^{i\pi}\end{pmatrix}\right] \circ \left[\begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\end{pmatrix} \otimes I\right] \circ \left[I \otimes \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{pmatrix}\right] \circ [I \otimes H \otimes I]$$

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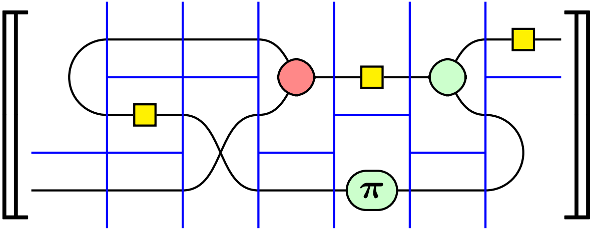

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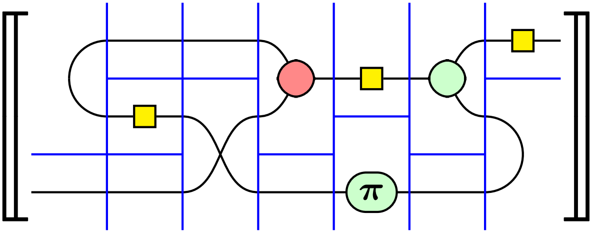

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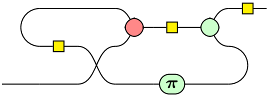
ZX-calculus III: The rules



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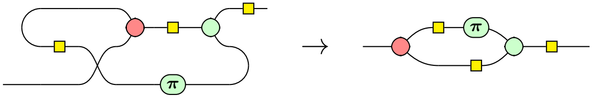

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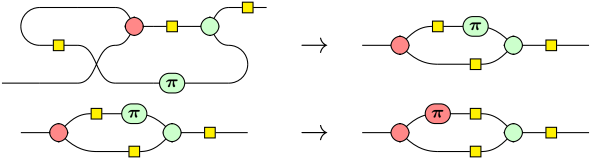

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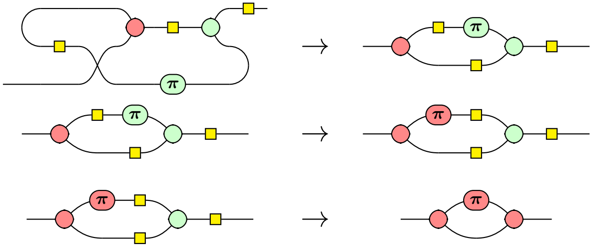

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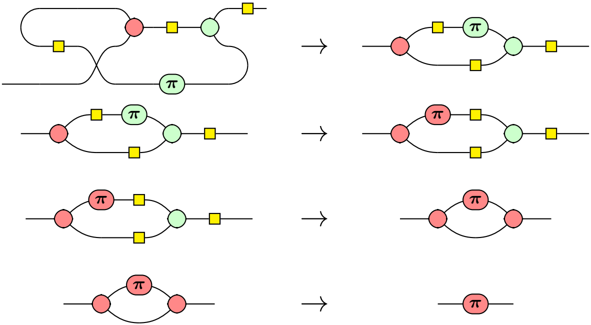

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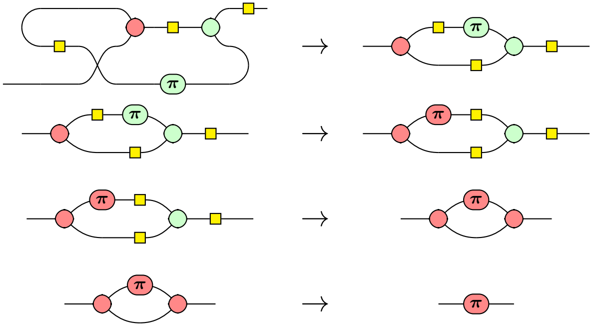
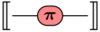
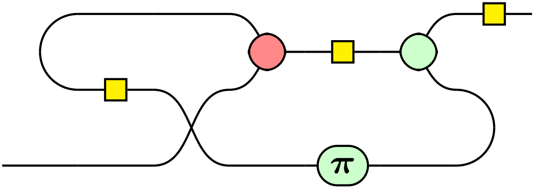




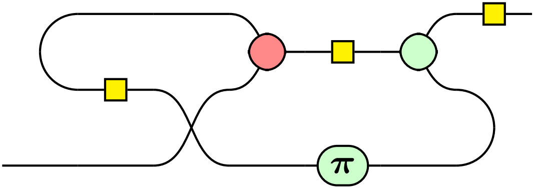
$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

ZX-calculus IV: Conclusion



The ZX-calculus provides:

ZX-calculus IV: Conclusion



The ZX-calculus provides:

- Intuitive graphical calculus.

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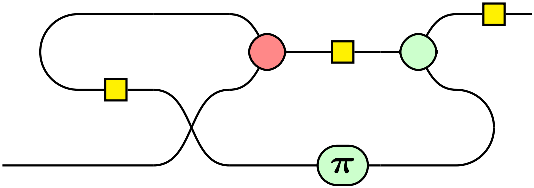

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- Complete equational theory for **any number** of qubits.

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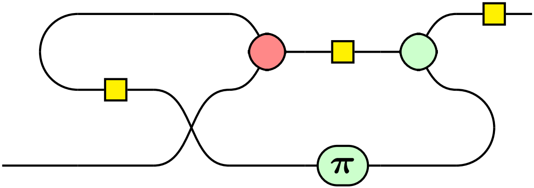

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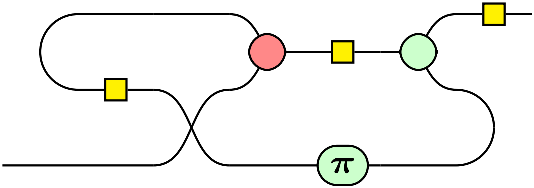

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- Intuitive graphical calculus.
- Complete equational theory for **any number** of qubits.
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We can be even more compact while scaling up the number of qubits!

$\mathcal{S}$ZX-calculus I: Divide and Gather

$$a_b \;\text{———}\; a_b$$

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$$a_b \;\text{——}\; a_b \qquad\qquad 1_2 \;\text{——}\; 1_2 \;\neq\; \begin{matrix} 1_1 & \text{——} & 1_1 \\ + & & + \\ 1_1 & \text{——} & 1_1 \end{matrix}$$

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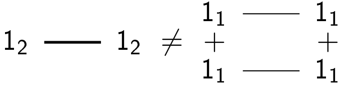

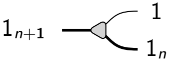

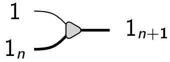

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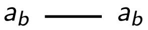

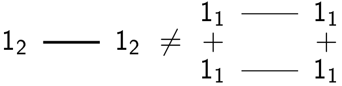

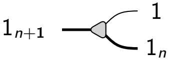

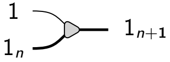

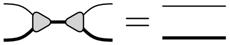

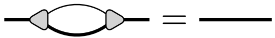

$\mathcal{S}$ZX-calculus II: The rewiring theorem

Theorem:

Let $\omega \in \mathbb{W}[a, b]$ and $\omega' \in \mathbb{W}[c, d]$:

$$\mathbb{W} \vdash \omega = \omega' \quad \Leftrightarrow \quad a = c \textit{ and } b = d$$

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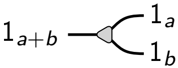

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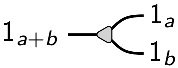
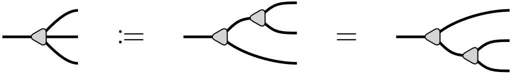
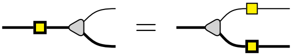




$\mathcal{S}$ZX-calculus III: The big generators



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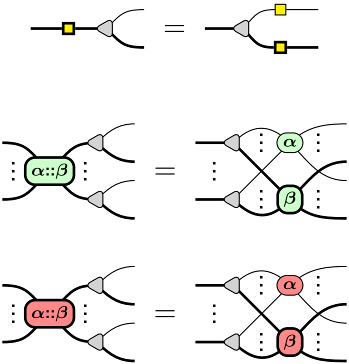

$\mathcal{S}$ZX-calculus IV: Completness

$$[\![—]\!]_s \ := \ (id_n, 1_n, 1_n)$$

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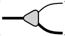

$$[\![\,\cdot\,]\!]_s \;:=\; (id_{n+1}, 1_{n+1}, 1{+}1_n)$$

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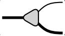
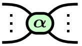
$[\![$  $]\!]_s \; := \; (id_{n+1}, 1_{n+1}, 1{+}1_n)$

$[\![$  $]\!]_s \; := \; \left(\sum_{x \in \{0,1\}^n} e^{ix \bullet \alpha} |x^k\rangle\langle x^\ell|, k_n, \ell_n\right)$

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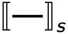
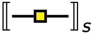
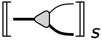
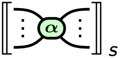
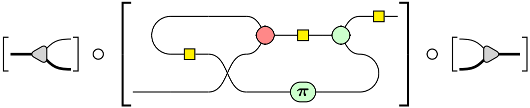
 $:= \; (id_n, 1_n, 1_n)$

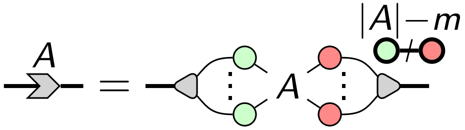
 $:= \; \left(\frac{1}{\sqrt{2}^n} \sum_{x,y \in \{0,1\}^n} (-1)^{x \bullet y} |y\rangle\langle x|, 1_n, 1_n\right)$

 $:= \; (id_{n+1}, 1_{n+1}, 1+1_n)$

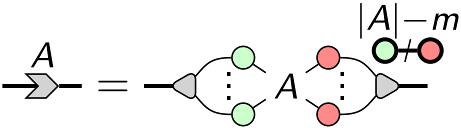
 $:= \; \left(\sum_{x \in \{0,1\}^n} e^{ix \bullet \alpha} |x^k\rangle\langle x^\ell|, k_n, \ell_n\right)$



$\mathcal{S}$ZX-calculus V: Matrices

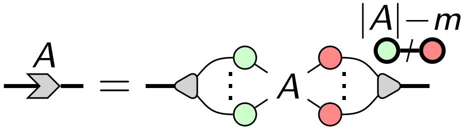


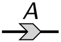
$\mathcal{S}$ZX-calculus V: Matrices



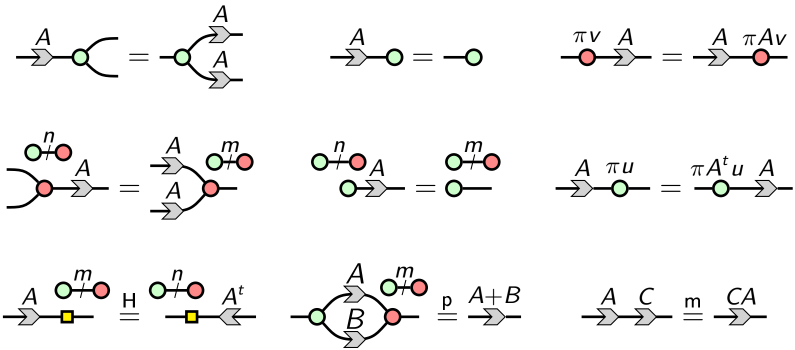
$$\forall A \in \mathbb{F}_2^{m\times n}, \left[\!\left[\xrightarrow{A} \right]\!\right]_s = (|x\rangle \mapsto |Ax\rangle, 1_n, 1_m)$$

$\mathcal{S}$ZX-calculus V: Matrices

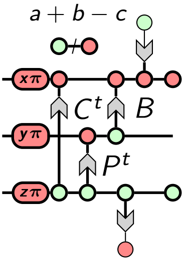


$\forall A \in \mathbb{F}_2^{m\times n},\ \left[\!\left[$  $\right]\!\right]_s = (|x\rangle \mapsto |Ax\rangle, 1_n, 1_m)$

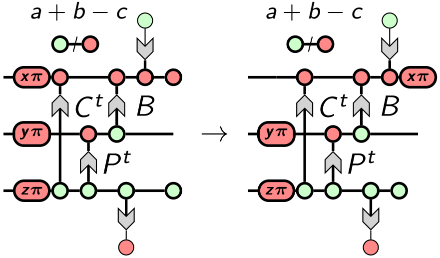
$\mathcal{S}$ZX-calculus VI: Properties of matrices



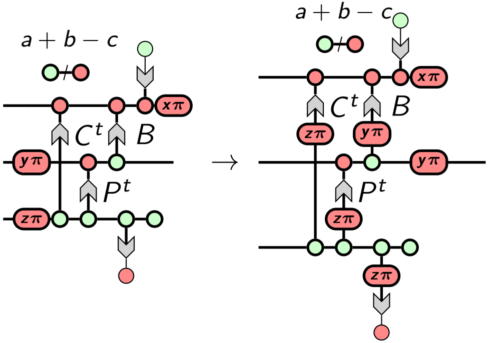
$\mathcal{S}$ZX-calculus VII: An application



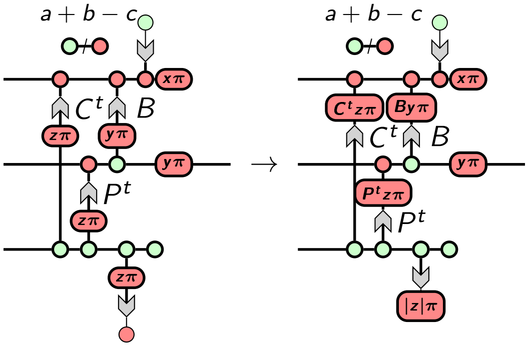
$\mathcal{S}$ZX-calculus VII: An application



$\mathcal{S}$ZX-calculus VII: An application



$\mathcal{S}$ZX-calculus VII: An application



$\mathcal{S}$ZX-calculus VII: An application