Numerical verification of lower bounds in trapped-ions quantum circuits synthesis

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Outline

Introduction :

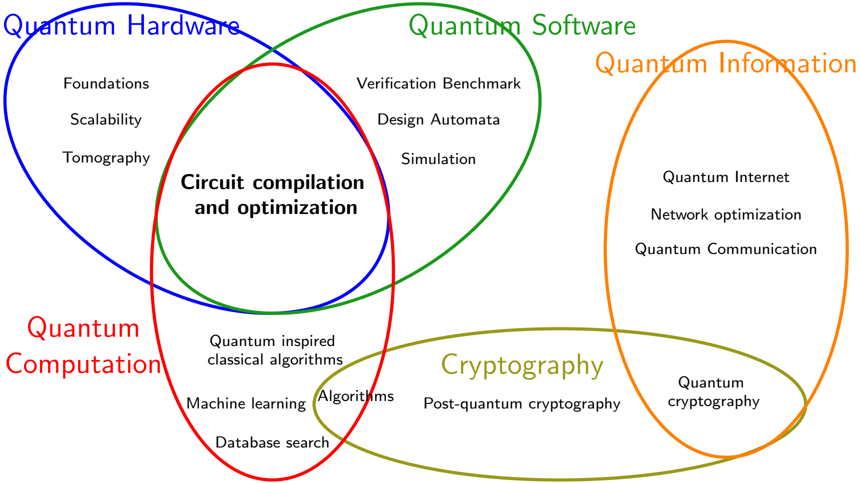
- Quantum computing, quantum circuits,
- Quantum circuit synthesis.

Main question : minimal number of gates required to implement any quantum operator ?

Our contribution :

- Lower bounds in the size of trapped-ions circuits,
- Synthesis framework using numerical optimization,
- Experimental results to confirm/infirm these lower bounds.

Fields of research



Quantum vs Classical computation

- Bit $= \{0, 1\} \rightarrow$ Qubit $\in \mathbb{C}^2$

$$|\psi\rangle = \alpha \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{False}} + \beta \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{True}}, |\alpha|^2 + |\beta|^2 = 1$$

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- All computations require linear algebra

<u>Tensor product</u> :

$$\underbrace{|\psi_3\rangle}_{\mathbb{C}^{2^{n+m}}} = \underbrace{|\psi_1\rangle}_{\mathbb{C}^{2^n}} \otimes \underbrace{|\psi_2\rangle}_{\mathbb{C}^{2^m}}$$

<u>Matrix-vector multiplication</u> :

$$|\psi_{t_2}\rangle = U\,|\psi_{t_1}\rangle \iff |\psi_{t_1}\rangle = U^H\,|\psi_{t_2}\rangle$$

$$U \in \mathcal{U}(2^n) = \{A \in \mathcal{M}_{2^n}(\mathbb{C}) | A^H A = I\}$$

Common quantum operators

- Examples of quantum operators on 1 qubit :

- $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = $ "NOT"
- $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$
- $R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$
- $R_x(\theta) = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix}$

- Entangling operator $\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

Quantum circuit

Quantum algorithm = quantum circuit = series of quantum gates

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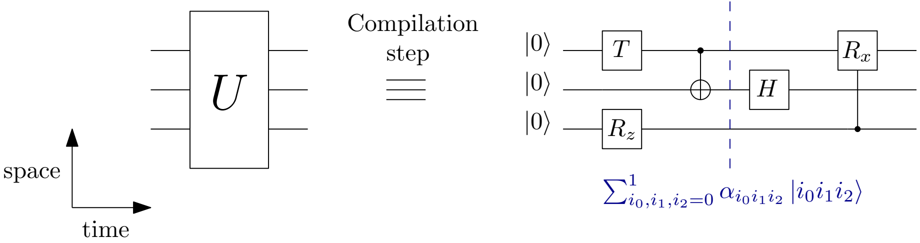
Space composition → tensor product
Time composition → matrix multiplication from the left



$$U = \Lambda_3(R_x) \times (I_2 \otimes H \otimes I_2) \times (CNOT \otimes I_2) \times (T \otimes I_2 \otimes R_z)$$

U is unknown to the hardware

Universality

Hardware constraint : small subset of gates natively available

Universality : a set of gates is universal if any operator can be synthesized using only these gates

Example : NAND (Toffoli) is universal in (reversible) classical computing

Famous universal sets for superconducting qubit processors (IBM, Google, Rigetti) :

Basic set : { CNOT, $\mathcal{U}(2)$ }
Elementary set : { CNOT, H, T }

Circuit synthesis

Problem : Given an n-qubits unitary operator $U \in \mathcal{U}(2^n)$,
find (using classical computer) a circuit that implements it with some given criteria:

- **lowest number of gates**,
- **specific set of gates (hardware constraints)**,
- allowed/forbidden extra memory (auxiliary qubits),
- *maximum approximation error* ($\|U - U_{synth}\|_F \leq \epsilon$),
- *minimal computation time*.

Special case : when U is a normalized column of a matrix, we talk about *state preparation*.

Lowest number of gates : Hard problem

[Shende, Bullock, Markov, IEEE, 2006]

$U \in \mathcal{U}(\mathbf{2^n})$: the complexity (in gate count) is exponential in n.
We assume here that U is generic.

Synthesis Algorithm	Number of qubits and gate counts 1	2	3	4	5	6	7	n
Original QR decomp. [3, 10]					——			$O(n^3 4^n)$
Improved QR decomp. [21]					——			$O(n4^n)$
Palindrome transform [2]					——			$O(n4^n)$
QR [33, Table I]	0	4	64	536	4156	22618	108760	$O(4^n)$
CSD [22, p. 4]	0	8	48	224	960	3968	16128	$4^n - 2 \times 2^n$
QSD ($l = 1$)	0	6	36	168	720	2976	12096	$(3/4) \times 4^n - (3/2) \times 2^n$
QSD ($l = 2$)	0	3	24	120	528	2208	9024	$(9/16) \times 4^n - (3/2) \times 2^n$
QSD ($l = 2$, optimized)	0	**3**	**20**	**100**	**444**	**1868**	**7660**	$(23/48) \times 4^n - (3/2) \times 2^n + 4/3$
Lower bounds [28, 29]	0	3	14	61	252	1020	4091	$\lceil \frac{1}{4}(4^n - 3n - 1) \rceil$

Many algebraic methods with an exponential number of gates

Hardware constraints : Trapped-ions circuits

Competing technology for the quantum supremacy :

- Circuits have high fidelity (= less sensitive to noise),
- qubits have a long decoherence time (= the processor can perform longer computations)

Different set of universal gates :

- local R_z gates
- global R_x gates (local R_x are applied to every qubit with the same angle θ)
- the entangling gate is the MS gate defined by

$$MS(\theta) = e^{-i\theta(\sum_{i=1}^{n} \sigma_x^i)^2/4} = H^{\otimes n} D(\theta) H^{\otimes n}$$

  with

$$D(\theta) = \text{diag}([e^{(n-Hamm(i))^2 \times \theta}]_{i=0..2^n-1})$$

MS gate is a **global** operation, acting on **all** qubits

Existing numerical methods

- BFGS algorithm for trapped-ions circuits
- machine learning techniques for photonic circuits
- genetic algorithms for IBM architecture

Yet no control on the optimality of the solution / Efficiency

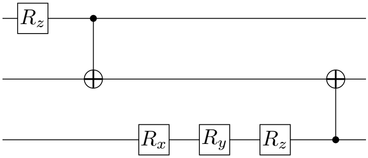
$\implies$ **can we use numerical methods to produce short circuits AND to provide some theoretical insights on the overall complexity ?**

- deriving lower bounds on the number of gates to synthesize any trapped-ions quantum circuits
- use numerical optimization to investigate the tightness of these bounds

Circuit topology

Circuit topology = abstraction of a circuit with unspecified parameterized gates : the degrees of freedom

A topology is summarized by a smooth function $f : \mathbb{R}^k \to \mathcal{U}(2^n)$



Example : for the circuit above, $f : \mathbb{R}^4 \to \mathcal{U}(8)$

Essential property : a topology is said to be *universal* if $\Im(f) = \mathcal{U}(2^n)$

$\Longrightarrow$ **how many gates a topology should at least contain to be universal ?**

Lower bounds for trapped-ions circuits (1/3)

The question was answered for $\{\mathcal{SU}(2), CNOT\}$ circuits
We do a similar reasoning for trapped-ions circuits, let

$$f : \mathbb{R}^k \to \mathcal{SU}(2^n),$$

- if $k < \dim(\mathcal{SU}(2^n)) = 4^n$ then $\Im(f)$ is of measure 0, thus

$$k \geq 4^n$$

- we derive a canonical form for trapped-ions circuits to get the maximum possible number of degrees of freedom in a topology

Lower bounds for trapped-ions circuits (2/3)

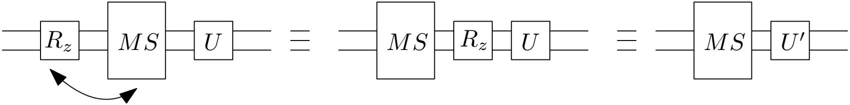
1) A circuit for trapped-ions has the layer decomposition

local gates | MS gate | ... | MS gate | local gates

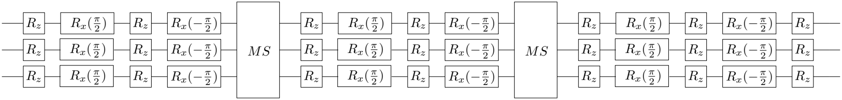
2) Any local gate can be decomposed

$$U = R_z(\alpha) \times R_x(-\pi/2) \times R_z(\beta) \times R_x(\pi/2) \times R_z(\gamma)$$

3)



Final canonical form :



Lower bounds for trapped-ions circuits (3/3)

- "At the left" of each MS gate we can have at most $2n$ DOF,
- we assume the MS are parameterized,
- the last layer of local gates can have at most $3n$ DOF,
- a global phase counts for one DOF.

Circuits on n qubits with k MS gates : $(2n+1)k + 3n + 1$ DOF

$$\#MS \geq \left\lceil \frac{4^n - 3n - 1}{2n+1} \right\rceil.$$

For state preparation :

$$\#MS \geq \left\lceil \frac{2^{n+1} - 2n - 2}{2n+1} \right\rceil.$$

Can we synthesize numerically any quantum operator with circuits of this size ?

Numerical optimization framework

We simplify the problem by focusing on one state only (= state preparation)
Given a state $|\psi\rangle$ and a topology f, the best circuit to implement $|\psi\rangle$ in f in the solution of

$$\arg\min_x \| f(x)\,|0\rangle - |\psi\rangle \| = \arg\min_x g(x)$$

where we choose $\|\cdot\|$ to be the euclidean norm, we have :

$$g(x) = 2 \times \left(1 - \Re\left(\langle 0|\, f(x)^\dagger\, |\psi\rangle\right)\right).$$

The gradient also has a useful analytical expression such that :
computing $g(x)$ and $\frac{\partial}{\partial x} g$ can be performed in 3 numerical simulations of the circuit

Experimental setup

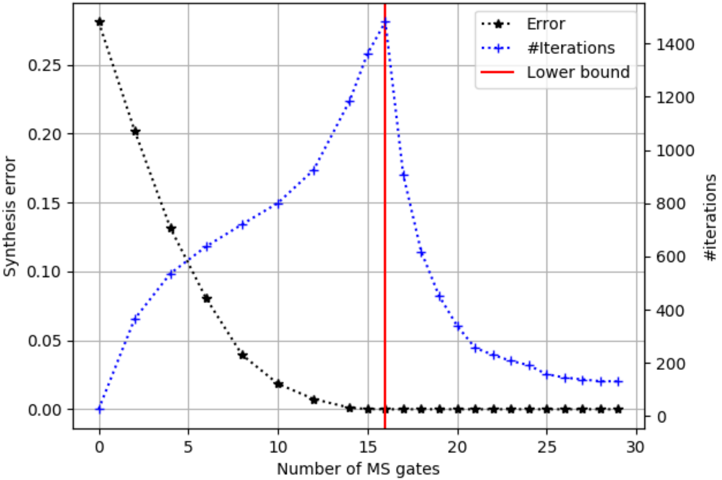
50 random unitaries on 4 qubits / 50 random states on 7 qubits.
Optimizations run for circuits with various number of MS gates.

We store for each circuit size :

- the maximum error encountered among the sample,
- the average number of iterations needed.

Core of the calculation in C with OpenMP, interface with Python and Scipy for the optimization.
Experiments achieved using a 24-core Intel Xeon(R) E7-8890 v4 processor at 2.4 GHz 24-cores.

Experimental results



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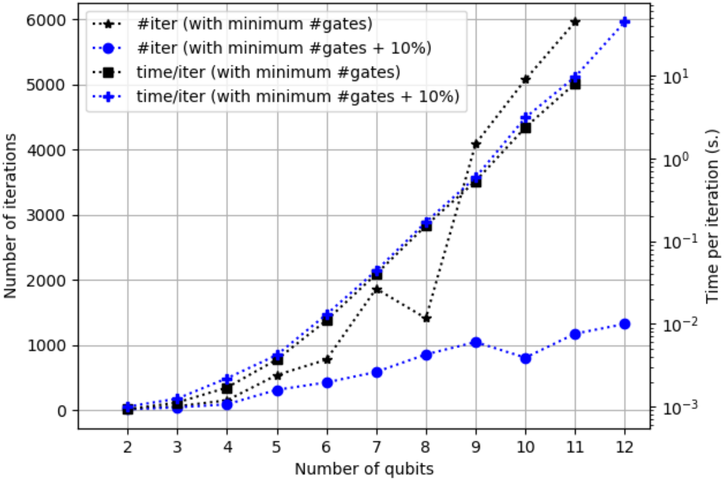

Conclusion and future work

During this talk :

- we introduced the main issues of quantum circuit synthesis,
- we derived lower bounds on the minimum size of a universal trapped-ions quantum circuit,
- we used a simple optimization framework such that we can:
  - ⋆ compute optimal circuits up to 4/7 qubits for unitaries/states,
  - ⋆ compute close to optimal circuits up to 6/12 qubits,
  - ⋆ confirm the tightness of the bounds.

Future work :

- using GPU and distributed computing to address larger problems,
- Hessian calculation for more complex optimizers,
- extension to $\{\mathcal{SU}(2), CNOT\}$ circuits.